A GENERALIZATION OF THE NORMAL HOLOMORPHIC FRAMES IN SYMPLECTIC MANIFOLDS

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Abstract. In this paper we give a generalization of the normal holomorphic frames in the symplectic manifolds and find conditions for the integrability of complex structures.

1. Introduction

One of the possible ways of studying symplectic manifolds is to fix a complex structure calibrated by the symplectic form and then to consider a structure which is the natural generalization of Kähler structure (sometimes in literature this structure is called almost-Kähler structure).
Moreover we can try to extend Kähler features in symplectic manifolds and find conditions on the Riemannian invariants (like the curvature e.g.) which force the integrability of the complex structure. These problems are object of research of many papers (see e.g. [1], [3], [4], [11]).
Starting from an idea of P. de Bartolomeis and A. Tomassini (see [6]), in this paper we generalize the normal holomorphic frames, characteristic of the Kähler manifolds, to the symplectic case (see theorem 1); we call these frames *generalized normal holomorphic frames*.
As application of the existence of generalized normal holomorphic frames we prove that if the (0,1)-part of the covariant derivative of the (1,2)-tensor

$$B(X,Y) = J(\nabla_X J)Y - (\nabla_{JX} J)Y$$

vanishes, then the complex structure J is integrable (see theorem 2).

The paper is organized as follows. After some preliminaries, in section 3 we give the proof of the existence of generalized normal holomorphic frames and in section 4 we prove theorem 2. Finally we give

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another sufficient condition for the integrability of J. Namely, we show that if the complex structure satisfies

$$(\nabla J)\nabla = 0\,,$$

then J is holomorphic.

2. Preliminaries

Let M be a $2n$-dimensional manifold: a *complex structure* on M is a smooth section J of the bundle $\mathrm{End}(TM)$ such that $J^2 = -Id$. The couple (M, J) is called a *complex manifold*.
A complex structure gives a natural split of $TM \otimes \mathbb{C}$ in $TM^{(1,0)} \oplus TM^{(0,1)}$, where $TM^{(1,0)}$ and $TM^{(0,1)}$ are the eigenspaces relatively to i and $-i$.
A Riemannian J-invariant metric g on a complex manifold (M, J) is said to be Hermitian and the triple (M, g, J) is by definition an Hermitian manifold. An Hermitian metric g induces a non-degenerate 2-form κ on M, given by

$$\kappa(J\cdot, \cdot) = g(\cdot, \cdot)\,.$$

It is well known the fundamental Hermitian relation:

$$2g((\nabla_X J)Y, Z) = d\kappa(X, JY, JZ) - d\kappa(X, Y, Z) + g(N_J(Y, Z), JX) \tag{1}$$

(see e.g. [9]), where ∇ is the Levi-Civita connection of g and N_J is the Nijenhuis tensor of J, i.e. the (1,2)-tensor defined on M by

$$N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]\,.$$

A holomorphic manifold M has a natural complex structure. A complex structure J is said to be integrable if it is induced by a holomorphic structure. In view of Newlander-Nirenberg theorem a complex structure is integrable if and only if N_J vanishes.

A *symplectic manifold* is a pair (M, κ), where κ is a non-degenerate closed 2-form (i.e. $\kappa^n \neq 0, \;\; d\kappa = 0$) .
A complex structure J on a symplectic manifold (M, κ) is said to be κ-calibrated if

1. $\kappa(J\cdot, J\cdot) = \kappa(\cdot, \cdot)$ (κ is J-invariant);
2. $g_J(\cdot, \cdot) := \kappa(J\cdot, \cdot)$ is a positive defined tensor.

Obviously g_J is a Hermitian metric on M. Let us denote by $\mathcal{C}_\kappa(M)$ the set of the κ-calibrated complex structures on M: it is well known that $\mathcal{C}_\kappa(M)$ is always a non-empty set (see e.g. [2]).

A *Kähler structure* on M is a blend of three components (g, J, κ), where κ is a symplectic structure, J is an integrable κ-calibrated complex structure and $g = g_J$ (for the general theory about Kähler manifolds we remand to [7], [8], [9] and [10]).
It's known that a holomorphic Hermitian manifold (M, g, J) is Kähler if and only if around every point o in M there exists a normal holomorphic coordinate system. Namely, if o is any point of M, there exists a coordinate system $\{z_1, \ldots, z_n\}$ around o such that, if $G_{ij} := g(\partial/\partial z_i, \partial/\partial \overline{z}_j)$, then

1. $G_{ij}(o) = \delta_{ij}$;
2. $dG_{ij}[o] = 0$.

A system of normal coordinates induces a (1,0)-frame $\{Z_1, \ldots, Z_n\}$ around o such that

$$\nabla_{Z_i} Z_k(o) = 0\,, \qquad \nabla_{\overline{Z}_i} Z_k(o) = 0\,,$$

for $1 \leq i, k \leq n$, where ∇ is the Levi-Civita connection of g. We call such frames *"normal holomorphic frames"*.

The aim of this paper is to construct a generalization of the normal holomorphic frames in the symplectic manifolds and find some non-obvious conditions which imply $N_J = 0$.

At first we note that, in the symplectic case, (1) reduces to

$$2g((\nabla_X J)Y, Z) = g(N_J(Y, Z), JX)\,; \tag{2}$$

so we have:

Corollary 2.1. *Let (M, κ) be a symplectic manifold, $J \in \mathcal{C}_\kappa(M)$, $g = g_J$ and ∇ the Levi-Civita connection of g.*
Then $\nabla_{\overline{Z}_1} Z_2$ is a (1,0)-vector field on M for any Z_1, Z_2 of type (1,0).

Proof. It's easy to check that:

$$N_J(Z_1, \overline{Z}_2) = 0\,, \tag{3}$$

for any Z_1, Z_2 $(1,0)$-vector fields.
Therefore, from the complex extension of (2), we have:

$$g((\nabla_{\overline{Z}_1} J)Z_2, Z_3) = 0\,,$$
$$g((\nabla_{\overline{Z}_1} J)Z_2, \overline{Z}_3) = 0.$$

Hence

$$0 = (\nabla_{\overline{Z}_1} J)Z_2 = \nabla_{\overline{Z}_1} JZ_2 - J\nabla_{\overline{Z}_1} Z_2 = i\nabla_{\overline{Z}_1} Z_2 - J\nabla_{\overline{Z}_1} Z_2\,.$$

$\square$

Now we have an obstruction to generalize normal holomorphic frames :

Corollary 2.2. *Let (M,κ) be a symplectic manifold, $J\in\mathcal{C}_\kappa(M)$ and $g=g_J$.*
Assume that for every point $o\in M$ there exists a local complex (1,0)-frame
$\{Z_1,\dots,Z_n\}$, around o, such that:

1. *$\nabla_{Z_i}\overline{Z}_k(o)=0$, $1\le i,k\le n$;*
2. *The real underlying frame $\{X_1,\dots,X_n\}$ satisfies*

$$\nabla_{X_i}X_k(o)=0\,,\;\nabla_{JX_i}X_k(o)=0\,,\;1\le i,k\le n\,,$$

then (g,J,κ) is a Kähler structure on M.

Proof. Let $\{Z_1,\dots,Z_n\}$ be a frame satisfying 1-2. Putting $Z_k=X_k-iJX_k$ and $Z_i=X_i-iJX_i$, by hypothesis 1 it follows

$$0=\nabla_{X_i-iJX_i}(X_k+iJX_k)(o)=\nabla_{X_i}X_k(o)+\nabla_{JX_i}JX_k(o)+i(\nabla_{X_i}JX_k(o)-\nabla_{JX_i}X_k(o))\,.$$

Then hypothesis 2 implies

$$0=\nabla_{JX_k}X_i(o)=\nabla_{X_k}JX_i(o)=\nabla_{JX_k}JX_i(o)\,. \tag{4}$$

Finally we have

$$\nabla_{Z_i}Z_k(o)=\nabla_{X_i}X_k(o)-\nabla_{JX_i}JX_k(o)+i(\nabla_{X_i}JX_k(o)+\nabla_{JX_i}X_k(o))$$

and so $\nabla_{Z_i}Z_k(o)=0$ which implies $N_J=0$. □

3. Construction of generalized normal holomorphic frames in symplectic manifolds

From [6] we have the following

Lemma 3.1 (Special frame on symplectic manifolds)**.** *Let (M,κ) be a 2n-manifold equipped with a non-degenerate 2-form, $J\in\mathcal{C}_\kappa(M)$ and $g=g_J$.*
The following facts are equivalent:

1. $\partial_J\kappa=\overline{\partial}_J\kappa=0$.
2. *For every $o\in M$ there exists a local complex (1,0)-frame $\{Z_1,\dots,Z_n\}$ around o such that:*
   a) $[Z_i,Z_j](o)=-(1/4)N_J[o](Z_i,Z_j)$, $[\overline{Z}_i,Z_j](o)=0$, $1\le r,s\le n$;
   b) *if $G_{ik}:=(g(Z_i,\overline{Z}_k))$, then $G_{ik}(o)=\delta_{ik}$, $dG_{ik}[o]=0$.*
3. *For every $o\in M$, there exists a local complex (1,0)-coframe $\{\zeta_1,\dots,\zeta_n\}$ around o such that:*

a) $\partial_J \zeta_r[o] = 0$, $\overline{\partial}_J \zeta_r[o] = 0$, $1 \leq r \leq n$;
b) *if* $H_{rs} := (g(\zeta_r, \overline{\zeta}_s))$, *then* $H_{rs}(o) = \delta_{rs}$, $dH_{rs}[o] = 0$.

We call a frame which satisfies a) and b) a *"special frame"*.
Now we have

Theorem 3.2 (Generalized normal holomorphic frames)**.** *Let* (M, κ) *be a symplectic manifold,* $J \in \mathcal{C}_\kappa(M)$*,* $g = g_J$ *and* o *an arbitrary point of* M*.*
Then there exists always a local complex (1,0)-frame, $\{W_1, \dots, W_n\}$ *around* o*, such that:*

1. $\nabla_{W_k} \overline{W}_i(o) = 0$, $1 \leq k, i \leq n$.
2. $\nabla_{W_k} W_i(o) \in TM^{(0,1)}$, $1 \leq k, i \leq n$.
3. *If* $G_{rs} := g(W_r, \overline{W}_s)$, *then:* $G_{rs}(o) = \delta_{rs}$, $dG_{rs}[o] = 0$.
4. $\nabla_{W_r} \nabla_{\overline{W}_k} W_i(o) = 0$, $1 \leq r, k, i \leq n$.

We call $\{W_1, \dots, W_n\}$ a *generalized normal holomorphic frame.*

Proof. Step 1:
Let $\{Z_1, \dots, Z_n\}$ be a special frame around o.

1. From the relation
$$0 = [Z_r, \overline{Z}_k](o) = \nabla_{Z_r} \overline{Z}_k(o) - \nabla_{\overline{Z}_k} Z_r(o)$$
it follows
$$\nabla_{Z_r} \overline{Z}_k(o) = \nabla_{\overline{Z}_k} Z_r(o)$$
and so by corollary 1
$$\nabla_{\overline{Z}_k} Z_r(o) = 0, \qquad \nabla_{Z_r} \overline{Z}_k(o) = 0 \quad 1 \leq r, k \leq n\,. \tag{5}$$
2. From (5), we have
$$\nabla_{Z_k} Z_i(o) \in T_o M^{(0,1)} \qquad 1 \leq k, i \leq n\,. \tag{6}$$
3. From (5) and corollary 1 we get that:
$$g(\nabla_{Z_r} \nabla_{\overline{Z}_k} Z_i, Z_s)(o) = -g(\nabla_{\overline{Z}_k} Z_i, \nabla_{Z_r} Z_s)(o) + Z_r g(\nabla_{\overline{Z}_k} Zi, Z_s)(o) = 0\,;$$
therefore $\nabla_{Z_r} \nabla_{\overline{Z}_k} Z_i(o)$ are vector fields of type (1,0) for any $1 \leq r, k, i \leq n$.
4. In a similar way:
$$g(\nabla_{Z_r} \nabla_{\overline{Z}_k} Z_i, \overline{Z}_s)(o) = Z_r g(\nabla_{\overline{Z}_k} Z_i, \overline{Z}_s)(o)\,.$$

So we can reduce to find a special frame $\{W_1, \dots, W_n\}$ around o which satisfies the relation
$$\partial_J (g(\nabla_{\overline{W}_k} W_i, \overline{W}_s))[o] = 0\,.$$

Step 2:
We can assume that a special frame satisfies at o

$$Z_i = \frac{\partial}{\partial z_i}\,,$$

for some complex local coordinates $\{z_1,\dots,z_n\}$ such that

$$z_i(o) = 0, \quad 1 \leq i \leq n\,.$$

Let $\{W_1,,\dots,W_n\}$ be the complex frame

$$W_i = Z_i - \sum_{h=1}^{n} A_{hi} Z_h\,, \tag{7}$$

where

$$A_{si} = \sum_{a,b=1}^{n} Z_a \Gamma_{\overline{b}i}^{s}(o) z_a \overline{z}_b$$

and

$$\Gamma_{\overline{b}i}^{s} := g(\nabla_{\overline{Z}_b} Z_i, \overline{Z}_s)\,.$$

It's easy to check that this frame is a special one and then, by the first step of the proof, it satisfies 1-3. So it's enough to show that $\{W_1,\dots,W_n\}$ satisfies 4.
Set $B_{si} = \delta_{si} - A_{si}$. Then we have

1. $B_{si}(o) = \delta_{si}$,
2. $\frac{\partial}{\partial z_a} B_{si}[o] = \frac{\partial}{\partial \overline{z}_b} B_{si}[o] = 0$,

then

$$g(\nabla_{\overline{W}_k} W_i, \overline{W}_s) = \sum_{a,b,c=1}^{n} \overline{B}_{ak}\overline{B}_{cs}\overline{Z}_a(B_{bi}) g(Z_b,\overline{Z}_c) + \overline{B}_{ak}\overline{B}_{cs} B_{bi} g(\nabla_{\overline{Z}_a} Z_b, \overline{Z}_c)$$

and

$$\begin{aligned}\partial_J(g(\nabla_{\overline{W}_k} W_i, \overline{W}_s))[o] =& \partial_J(\Gamma_{\overline{k}i}^{s})[o] + \sum_{a=1}^{n} \overline{Z}_a(B_{si})\partial_J(\overline{B}_{ak})[o] + \\ & \sum_{b=1}^{n} \overline{Z}_k(B_{si})\partial_J(\overline{B}_{bs})[o] + \partial_J(\overline{Z}_k(B_{si}))[o]\,.\end{aligned}$$

Therefore

$$\partial_J(g(\nabla_{\overline{W}_k} W_i, \overline{W}_s))[o] = \partial_J(\overline{Z}_k(B_{si}))[o] + \partial_J(\Gamma_{\overline{k}i}^{s})[o]\,.$$

Hence we have

$$
\begin{aligned}
-\partial_J(\overline{Z}_k(B_{si}))[o] = \sum_{a,b=1}^{n} \partial_J(\overline{Z}_k(Z_a\Gamma_{\overline{b}i}^{s}(o)z_a\overline{z}_b))[o] \\
+\sum_{a,b=1}^{n} Z_a\Gamma_{\overline{b}i}^{s}(o)\partial_J(\overline{Z}_k(z_a\overline{z}_b))[o] = \\
=\sum_{a,b=1}^{n} Z_a\Gamma_{\overline{b}i}^{s}(o)\partial_J(\overline{Z}_k(z_a)\overline{z}_b + \overline{Z}_k(\overline{z}_b)z_a)[o]\,.
\end{aligned}
$$

Finally we have

$$
Z_l(\overline{Z_k}(z_a)\overline{z_b} + \overline{Z_k}(\overline{z}_b)z_a)[o] = \delta_{kb}\delta_{la}\,.
$$

Therefore we obtain

$$
\partial_J(\overline{Z}_k(B_{si}))[o] = -\partial_J(\Gamma_{\overline{k}i}^{s})[o]
$$

which concludes the proof. □

Remark: In the previous lemma we can require

$$
\nabla_{W_k}\nabla_{W_j}\overline{W}_i(o) = 0
$$

instead of

$$
\nabla_{W_k}\nabla_{\overline{W}_j}W_i(o) = 0\,,
$$

but in general we can't require that the two conditions hold simultaneously.

Remark: If M is Kähler, then the vector fields $\nabla_{W_i}W_j$ are globally of (1,0)-type and then they vanish at o. Therefore in the Kähler case generalized normal holomorphic frames are normal holomorphic frames.

4. Integration of complex structures calibrated by symplectic forms

In this section we apply the construction of the generalized normal holomorphic frames (given in section 3) in order to find integrability conditions for calibrated complex structures.
Let (M, g, J) be a Hermitian manifold; let us denote by B the (2,1)-tensor defined by

$$
B(X, Y) = J(\nabla_X J)Y - (\nabla_{JX}J)Y\,.
$$

It is easy to check that B satisfies

1. $B(X, Y) - B(Y, X) = -N_J(X, Y)$;
2. $B(Z_1, Z_2) = 2iJ\nabla_{Z_1}Z_2 + 2\nabla_{Z_1}Z_2$;

3. $B(Z_1, Z_2) \in TM^{(0,1)}$;
4. $B(Z_1, \overline{Z}_2) = 0$,

for every X, Y in TM and Z_1, Z_2 in $TM^{(1,0)}$.
Therefore, we have:

Lemma 4.1. *Let (M, g, J) be a Hermitian manifold, then:*

$$N_J = 0 \iff B = 0\,.$$

In the symplectic case, it is natural to ask if the condition $B = 0$ can be weaken. An answer to this question is given by the following

Theorem 4.2. *Let (M, κ) be a symplectic manifold, $J \in \mathcal{C}_\kappa(M)$ and $g = g_J$.*

If $\nabla'' B = 0$ then M is a Kähler manifold,

where ∇'' is the (0,1)-part of the Levi-Civita connection (i.e $\nabla''_Z := \nabla_{Z^{(0,1)}}$).

Proof. Let o be an arbitrary point of M and let $\{Z_1, \ldots, Z_n\}$ be a generalized normal holomorphic frame around o.
We have

$$B(Z_i, Z_k)(o) = 4\nabla_{Z_i} Z_k(o), \;\; 1 \leq i, k \leq n\,.$$

By the properties of B we obtain

$$Z_l g(B(Z_i, Z_k), \overline{Z}_r)[o] = 0, \;\; 1 \leq l, i, k, r \leq n\,.$$

Then

$$\begin{aligned}
0 =& \overline{Z}_l g(B(Z_i, Z_k), \overline{Z}_r)[o] = \\
=& g(\nabla_{\overline{Z}_l}(B(Z_i, Z_k)), \overline{Z}_r)(o) + g(B(Z_i, Z_k), \nabla_{\overline{Z}_l}\overline{Z}_r)(o) = \\
=& g((\nabla_{\overline{Z}_l}B)(Z_i, Z_k), \overline{Z}_r)(o) + g(B(Z_i, Z_k), \nabla_{\overline{Z}_l}\overline{Z}_r)(o) = \\
=& g(B(Z_i, Z_k), \nabla_{\overline{Z}_l}\overline{Z}_r)(o) = \\
=& 4g(\nabla_{Z_i} Z_k, \nabla_{\overline{Z}_l}\overline{Z}_r)(o)\,,
\end{aligned}$$

so in particular we obtain

$$g(\nabla_{Z_i} Z_k, \nabla_{\overline{Z}_i}\overline{Z}_k)(o) = 0\,.$$

Hence we get $\nabla_{Z_i} Z_k(o) = 0$ for $1 \leq i, k \leq n$, which implies $N_J = 0$. □

Remark: In the previous theorem it's enough to require

$$g((\nabla_{\overline{Z}_1}B)(Z_1, Z_2), \overline{Z}_2) = 0$$

for every Z_1, Z_2 (1,0)-fields.

Now we give an integrability condition in terms of curvature.

Lemma 4.3. *Let (M,κ) be a symplectic manifold, $J \in \mathcal{C}_\kappa(M)$, $Ag = g_J$. Let R be the curvature of g. These facts are equivalent:*

1. $\nabla'' B = 0\,;$
2. $R(\overline{Z},W)H = -i\nabla_{\overline{Z}}J\nabla_W H + iJ\nabla_W\nabla_{\overline{Z}}H + iJ\nabla_{\nabla_{\overline{Z}}W}H + \nabla_{\nabla_W\overline{Z}}H\,, \forall Z,W,H \in TM^{(1,0)}\,;$
3. $(\nabla_{\overline{Z}}J)(\nabla_W H) = 0 \quad \forall Z,W,H,S \in TM^{(1,0)}\,.$

Proof. $1 \iff 2$: Let $Z,W,H \in TM^{(1,0)}$, then

$$\begin{aligned}
\frac{1}{2}(\nabla_{\overline{Z}}B)(W,H) =& \frac{1}{2}\{\nabla_{\overline{Z}}(B(W,H)) - B(\nabla_{\overline{Z}}W,H) - B(W,\nabla_{\overline{Z}}H)\} = \\
=& \nabla_{\overline{Z}}\nabla_W H + i\nabla_{\overline{Z}}J\nabla_W H - \nabla_{\nabla_{\overline{Z}}W}H - iJ\nabla_{\nabla_{\overline{Z}}W}H \\
& - \nabla_W\nabla_{\overline{Z}}H - iJ\nabla_W\nabla_{\overline{Z}}H = \\
=& [\nabla_{\overline{Z}},\nabla_W]H + i\nabla_{\overline{Z}}J\nabla_W H - \nabla_{\nabla_{\overline{Z}}W}H - iJ\nabla_W\nabla_{\overline{Z}}H \\
& - iJ\nabla_{\nabla_{\overline{Z}}W}H = \\
=& R(\overline{Z},W)H + i\nabla_{\overline{Z}}J\nabla_W H - iJ\nabla_W\nabla_{\overline{Z}}H \\
& - iJ\nabla_{\nabla_{\overline{Z}}W}H - \nabla_{\nabla_W\overline{Z}}H\,,
\end{aligned}$$

and so

$$\begin{aligned}
\nabla''B = 0 \iff R(\overline{Z},W)H =& -i\nabla_{\overline{Z}}J\nabla_W H + iJ\nabla_W\nabla_{\overline{Z}}H \\
& + iJ\nabla_{\nabla_{\overline{Z}}W}H + \nabla_{\nabla_W\overline{Z}}H \\
\forall Z,W,H \in TM^{(1,0)}.
\end{aligned}$$

$1 \iff 3$: Let o be a point in M and let $\{Z_1,\dots,Z_n\}$ be a generalized normal holomorphic frame around o. At the point o we have

$$\begin{aligned}
g((\nabla_{\overline{Z}_i}J)\nabla_{Z_j}Z_k,\overline{Z}_r) =& -g(J\nabla_{\overline{Z}_i}\nabla_{Z_j}Z_k,\overline{Z}_r) + g(\nabla_{\overline{Z}_i}J\nabla_{Z_j}Z_k,\overline{Z_r}) = \\
& -ig(\nabla_{\overline{Z}_i}\nabla_{Z_j}Z_k,\overline{Z}_r) + i\overline{Z}_i g(\nabla_{Z_j}Z_k,\overline{Z}_r) \\
& + ig(\nabla_{Z_j}Z_k,\nabla_{\overline{Z}_i}\overline{Z}_r) = \\
=& 2ig(\nabla_{Z_j}Z_k,\nabla_{\overline{Z}_i}\overline{Z}_r)
\end{aligned}$$

for every $1 \le i,j,k,r \le n$. It is enough to prove $1 \iff 3$. □

Recall that an Hermitian manifold (M,g,J) is Kähler if and only if J is parallel, i.e.

$$\nabla J = 0\,.$$

As application of Lemma 3 we have the following

Theorem 4.4. *Let (M,κ) be a symplectic manifold, $J \in \mathcal{C}_\kappa(M)$ and $g = g_J$. If*

$$(\nabla J)\nabla = 0\,,$$

then (g, J, κ) is a Kähler structure on M.

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